Regularity of a inverse problem for generic parabolic equations

Nikolai Dokuchaev

Department of Mathematics, Trent University, Ontario, Canada

February 1, 2008

Abstract

The paper studies some inverse boundary value problem for simplest parabolic equations such that the homogenuous Cauchy condition is ill posed at initial time. Some regularity of the solution is established for a wide class of boundary value inputs.

Key words: inverse problems, parabolic equations, boundary value conditions, regularity, frequency domain, Hardy spaces.

AMS 2000 classification : 35K20, 35Q99, 32A35, 47A52.

Parabolic equations such as heat equations have fundamental significance for natural sciences, and various boundary value problems for them were widely studied including well-posed problems as well as the so-called inverse and ill-posed problems that often are very significant for applications (see, e.g., Beck (1985)). The present paper investigates a inverse boundary value problem on semi-plane for homogenuous parabolic equations with homogenuous Cauchy condition at initial time and with Dirichlet condition on the boundary of semi-plane. The parabolic equation is the equation of a backward type that is usually solvable with Cauchy condition at terminal time. However, we consider this equation with Cauchy condition at initial time so that the problem is an inverse problem. One may think that these problems are always ill-posed in the sense that there is no regularity of solutions such as prior estimates for the solution via some norms of free terms (see, e.g., Beck (1985), Tikhonov and Arsenin (1977)). We found a class of very generic parabolic equation with constant coefficients and with certain

sign of the drift coefficient (i.e., the coefficient for first derivative) such that the inverse problem has some regularity. More precisely, we found a wide enough class of inputs in the boundary value condition on the boundary of the semi-plane that ensures regularity in a form of prior energy type estimates. This class of inputs is everywhere dense in the class of L_2-integrable functions; it includes differentiable functions.

1 The problem setting

Let us consider the following boundary value problem on semi-plane:

$$\begin{aligned}
&a\frac{\partial u}{\partial t}(x,t)+\frac{\partial^2 u}{\partial x^2}(x,t)+b\frac{\partial u}{\partial x}(x,t)+cu(x,t)=0, \quad x>0,\ t>0,\\
&u(x,0)\equiv 0, \quad x>0,\\
&k_0u(0,t)+k_1\frac{\partial u}{\partial x}(0,t)\equiv g(t), \quad t>0.
\end{aligned} \tag{1}$$

Here $x>0$, $t>0$, and $a,b,c,k_0,k_1\in\mathbf{R}$ are constants.

We assume that

$$a>0, \quad b>0, \qquad k_0^2+k_1^2>0, \quad k_0k_1\le 0. \tag{2}$$

The assumption that $a>0$ and the presence of the initial condition at $t=0$ makes problem (1) an inverse problem (see, e.g., Beck (1985), Tikhonov and Arsenin (1977)).

We assume that $u(x,t)\equiv 0$ and $g(t)\equiv 0$ for $t<0$.

Let Γ denotes the set of all functions $g:\mathbf{R}\to\mathbf{R}$ such that $g(t)=0$ for $t<0$ and with finite norm

$$\|g\|_{W_2^1(\mathbf{R})}\triangleq\|g\|_{L_2(\mathbf{R})}+\left\|\frac{\partial g}{\partial t}\right\|_{L_2(\mathbf{R})}.$$

Remark 1 *Functions $g\in\Gamma$ are continuous on $\mathbf{R}$ and vanising on $t<0$ since $dg(t)/dt\in L_2(\mathbf{R})$. For instance, $g(t)=e^{-t}\sin t\in\Gamma$, but $g(t)=e^{-t}\cos t\notin\Gamma$.*

Let $D\triangleq\mathbf{R}\times\mathbf{R}^+$. Let $\mathcal{W}$ be the space of the functions $v=v(x,t):\mathbf{R}\times\mathbf{R}^+\to\mathbf{R}$ such that $v(x,t)\equiv 0$ for $t<0$ and with finite norm

$$\|v\|_{\mathcal{W}}\triangleq\|v\|_{L_2(D)}+\left\|\frac{\partial u}{\partial x}\right\|_{L_2(D)}+\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2(D)}+\left\|\frac{\partial v}{\partial t}\right\|_{L_2(D)}.$$

The class $\mathcal{W}$ is such that all the equations presented in problem (1) are well defined for any $u \in \mathcal{W}$. Let us show this. If $v \in \mathcal{W}$, then, for any $t_* > 0$, we have that $v|_{\mathbf{R}^+ \times [0,t_*]} \in C([0,t_*], L_2(\mathbf{R}^+))$ as a function of $t \in [0,t_*]$. Hence the initial condition at time $t = 0$ is well defined as an equality in $L_2(\mathbf{R}^+)$. Further, for any $x_* > 0$, we have that $v|_{[0,x_*]\times\mathbf{R}^+} \in C([0,x_*], L_2(\mathbf{R}^+))$ and $\frac{\partial v}{\partial x}\Big|_{[0,x_*]\times\mathbf{R}^+} \in C([0,x_*], L_2(\mathbf{R}^+))$ as functions of $x \in [0,x_*]$. Hence the functions $g_0(t) \triangleq v(0,t)$, $g_1(t) \triangleq \frac{du}{dx}(x,t)|_{x=0}$ are well defined as elements of $L_2(\mathbf{R}^+)$, and the boundary value condition at $x = 0$ is well defined as an equality in $L_2(\mathbf{R}^+)$.

2 The main result

Theorem 1 *Let condition (2) be satisfied, and let*

$$\mu \triangleq b^2/4 < c. \tag{3}$$

Then there exists a unique solution $u(x,t)$ in the class $\mathcal{W}$ of problem (1) in the domain D for any $g \in \Gamma$. Moreover, there exists a constant $C = C(a,b,c,k_0,k_1)$ such that

$$\|u\|_{\mathcal{W}} \leq C \|g\|_{W_2^1(\mathbf{R})}. \tag{4}$$

Remark 2 *It will be seen from the proof that it is crucial that $u(x,0) \equiv 0$ in the Cauchy condition and that the parabolic equation is homogenuous. We cannot extend the result for non-zero initial conditions or non-zero free term in the parabolic equation.*

The following theorem shows that assumption (3) is not really restrictive if we are not interested in the properties of the solutions for $T \to +\infty$, as can happen if we deal with solutions a finite time interval.

Theorem 2 *Let condition (2) holds, but (3) does not hold. Let M be such that $g(t)e^{-Mt} \in \Gamma$ and $b^2/4 < c + M$. Then problem (1) has a unique solution u such that $u_M \in \mathcal{M}$, where $u_M(x,t) \triangleq e^{-Mt}u(x,t)$.*

Proof of Theorem 1. Let $\mathbf{R}^+ \triangleq [0,+\infty)$, $\mathbf{C}^+ \triangleq \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$. For $v \in L_2(\mathbf{R})$, we denote by $\mathcal{F}v$ and $\mathcal{L}v$ the Fourier and the Laplace transforms respectively

$$V(i\omega) = (\mathcal{F}v)(i\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\omega t} v(t)dt, \quad \omega \in \mathbf{R}, \tag{5}$$

$$V(p) = (\mathcal{L}v)(p) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-pt} v(t) dt, \quad p \in \mathbf{C}^+. \tag{6}$$

Let H^r be the Hardy space of holomorphic on $\mathbf{C}^+$ functions $h(p)$ with finite norm $\|h\|_{H^r} = \sup_{k>0} \|h(k+i\omega)\|_{L^r(\mathbf{R})}$, $r \in [1, = \infty]$ (see, e.g., Duren (1970)).

Let $u \in \mathcal{W}$ be a solution of (1). Set $g_0(t) \triangleq u(0,t)$, $g_1(t) \triangleq \frac{du}{dx}(x,t)|_{x=0}$. As was discussed above, the functions g_k are well defined as elements of $L_2(\mathbf{R}^+)$.

Let $G \triangleq \mathcal{L}g$. We have that $G \in H^2$. Let $G_k(p) \triangleq \mathcal{L}g_k$ and $U \triangleq \mathcal{L}u$ be defined for $p \in \mathbf{C}^+$. They are well defined since $u \in \mathcal{W}$; in addition, $G_k \in H^2$.

For functions $V : \mathbf{R}^+ \times \bar{\mathbf{C}}^+ \to \mathbf{C}$, where $\bar{\mathbf{C}}^+ = \{z : \ \operatorname{Re} z \geq 0\}$, we introduce semi-norms

$$\|V\|_{L_{22}} \triangleq \left(\int_{\mathbf{R}^+} dx \int_{\mathbf{R}} |V(x, i\omega|^2 d\omega \right)^{1/2},$$
$$\|V\|_{\mathcal{W}^*} \triangleq \|V\|_{L_{22}} + \left\|\frac{\partial V}{\partial x}\right\|_{L_{22}} + \left\|\frac{\partial^2 V}{\partial x^2}\right\|_{L_{22}} + \|\frac{\partial V}{\partial t}\Big\|_{L_{22}}.$$

and norms

$$\|V\|_{L_{22}^H} \triangleq \left(\int_{\mathbf{R}^+} \|V(x,\cdot)\|_{H^2}^2 dx \right)^{1/2},$$
$$\|V\|_{\mathcal{H}} \triangleq \|V\|_{L_{22}^H} + \left\|\frac{\partial V}{\partial x}\right\|_{L_{22}^H} + \left\|\frac{\partial^2 V}{\partial x^2}\right\|_{L_{22}^H} + \left\|\frac{\partial V}{\partial t}\right\|_{L_{22}^H}.$$

Instead of (1), consider the problem

$$apU(x,p) + \frac{\partial^2 U}{\partial x^2}(x,p) + b\frac{\partial U}{\partial x}(x,p) + cU(x,t) = 0, \quad x > 0,$$
$$k_0 U(0,p) + \frac{\partial U}{\partial x}(0,p) \equiv G(p), \quad p \in \mathbf{C}^+ \tag{7}$$

subject to the following condition

$$U(x,\cdot), \frac{\partial U}{\partial x}(x,\cdot), \frac{\partial^2 U}{\partial x^2}(x,\cdot) \in H^2 \quad \text{for a.e} \quad x > 0, \qquad \|U\|_{\mathcal{H}} < +\infty. \tag{8}$$

Let $\lambda_k = \lambda_k(p)$ be the roots of the equation $\lambda^2 + b\lambda + (c + ap) = 0$ defined for $p \in \mathbf{C}^+$ as $\lambda_1 \triangleq -b/2 - \sqrt{\mu - ap}$ and $\lambda_2 \triangleq -b/2 + \sqrt{\mu - ap}$, where $\mu = b^2/4 - c < 0$. We mean the branch of the square root such that $\operatorname{Arg} \sqrt{\mu - ap} \in [-\pi/2, +\pi/2]$ and $\operatorname{Re} \sqrt{\mu - ap} \geq 0$. Under these assumptions, the function $\sqrt{\mu - ap}$ is holomorphic and does not have zeros in $\mathbf{C}^+$. We have that

$$\operatorname{Re} \lambda_1(p) \leq -\frac{b}{2}, \quad p \in \mathbf{C}_+,$$
$$\exists \delta > 0, \ \omega_* > 0 : \ \operatorname{Re} \lambda_2(i\omega) > \delta \quad \text{if} \quad \omega \in \mathbf{R}, \ |\omega| \geq \omega_*. \tag{9}$$

In addition, we have that the functions $\lambda_k(p)$ are holomorphic in $\mathbf{C}^+$, and

$$(\lambda_1(p)-\lambda_2(p))^{-1}\in H^\infty,\quad \lambda_k(p)(\lambda_1(p)-\lambda_2(p))^{-1}\in H^\infty,\quad k=1,2,$$
$$(k_0+k_1\lambda_1(p))^{-1}\in H^\infty,\quad \lambda_1(k_0+k_1\lambda_1(p))^{-1}\in H^\infty. \tag{10}$$

The last two statements here follow from (2). Let

$$N \triangleq \left\|\frac{1}{\lambda_1-\lambda_2}\right\|_{H^\infty}+\sum_{k=1,2}\left\|\frac{\lambda_k}{\lambda_1-\lambda_2}\right\|_{H^\infty}+\left\|\frac{1}{k_0+k_1\lambda_1}\right\|_{H^\infty}+\left\|\frac{\lambda_1}{k_0+k_1\lambda_1}\right\|_{H^\infty}.$$

It can be seen also that the functions $e^{x\lambda_k(p)}$ are holomorphic in $\mathbf{C}^+$ for any $x>0$.

For any $x>0$, the unique solution of (7) is

$$U(x,p) \;=\; \frac{1}{\lambda_1-\lambda_2}\Bigl((G_1(p)-\lambda_2G_0(p))e^{\lambda_1 x}+(G_1(p)-\lambda_1G_0(p))e^{\lambda_2 x}\Bigr).$$

This can be derived, for instance, using Laplace transform method applied to linear ordinary differential equation (7), and having in mind that

$$\frac{1}{\lambda^2+b\lambda+c-ap}=\frac{1}{(\lambda-\lambda_1)(\lambda-\lambda_2)}=\frac{1}{\lambda_1-\lambda_2}\left(\frac{1}{\lambda-\lambda_1}-\frac{1}{\lambda-\lambda_2}\right),$$
$$\frac{\lambda}{\lambda^2+b\lambda+c-ap}=\frac{\lambda}{(\lambda-\lambda_1)(\lambda-\lambda_2)}=\frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda-\lambda_1}-\frac{\lambda_2}{\lambda-\lambda_2}\right).$$

Let represent U as $U(x,p)=U_1(x,p)+U_2(x,p)$, where

$$U_1(x,p)=e^{\lambda_1 x}J_1(p),\qquad J_1(p)=\frac{1}{\lambda_1-\lambda_2}(G_1(p)-\lambda_2G_0(p)),$$
$$U_2(x,p)=e^{\lambda_2 x}J_2(p),\qquad J_2(p)=\frac{1}{\lambda_1-\lambda_2}(G_1(p)-\lambda_1G_0(p)).$$

By (9), the fact that $u\in\mathcal{W}$ implies that (8) holds. Let us show that

$$\|U_1\|_{L_{22}}<+\infty. \tag{11}$$

By (9), $|e^{x\lambda_1(p))}|\le e^{-bx/2}<1$, $p\in\mathbf{C}^+$. It follows that

$$\|e^{x\lambda_1(p)}J_1(p)\|_{H^2}\le\sup_{p\in\mathbf{C}^+}|e^{x\lambda_1(p))}|\|J_1\|_{H^2}\le e^{-bx/2}\|J_1\|_{H^2}\le Ne^{-bx/2}\sum_{k=0,1}\|G_k\|_{H^2}.$$

Then (8) and (11) imply that

$$\|U_2\|_{L_{22}}<+\infty. \tag{12}$$

Further,

$$+\infty > \|U_2\|^2_{L_{22}} = \int_{\mathbf{R}^+} dx \int_{\mathbf{R}} |U_2(x,i\omega)|^2 d\omega = \int_{\mathbf{R}^+} dx \int_{\mathbf{R}} |e^{\lambda_2(i\omega)x} J_2(i\omega)|^2 d\omega$$
$$= \int_{\mathbf{R}^+} dx \int_{\mathbf{R}} e^{\mathrm{Re}\,\lambda_2(i\omega)x} |J_2(i\omega)|^2 d\omega \geq \int_{\mathbf{R}^+} dx e^{\delta x} \int_{\omega:|\omega|\geq\omega_*} |J_2(i\omega)|^2 d\omega.$$

Note that the $J_2(i\omega)$ is vanishing on $\{\omega : |\omega| \geq \omega_*\}$. Since $J_2 \in H^2$, it follows that

$$J_2 = (G_1(p) - \lambda_1 G_0(p)) e^{\lambda_2 x} \equiv 0,$$

i.e.,

$$G_1(p) = \lambda_1 G_0(p).$$

Remind that $k_0 G_0(p) + k_1 G_1(p) = G(p)$. It follows that

$$k_0 G_0(p) + k_1 \lambda_1 G_0(p) = G(p), \qquad G_0(p) = (k_0 + k_1 \lambda_1(p))^{-1} G(p),$$
$$J_1(p) = G_0(p), \qquad U(x,p) = U_1(x,p) = e^{\lambda_1 x} G_0(p).$$

Let us estimate $\|U\|_{\mathcal{H}}$.

By (9), $|e^{x\lambda_1(p))}| \leq e^{-bx/2} < 1$. It follows that

$$\|p^m e^{x\lambda_1(p)} G_0(p)\|_{H^2} \leq e^{-bx/2} \|p^m G_0(p)\|_{H^2}$$
$$\leq e^{-bx/2} \|(k_0 + k_1\lambda_1(p))^{-1}\|_{H^\infty} \|p^m G\|_{H^2} \leq N e^{-bx/2} \|p^m G\|_{H^2}, \quad m = 0, 1.$$

It follows from the above estimate that

$$\|p^m U\|_{L^H_{22}} \leq N C_1 \|p^m G\|_{H^2}, \quad m = 0, 1. \tag{13}$$

Further, we have that

$$\frac{\partial U}{\partial x}(x,p) = G_0(p) \lambda_1 e^{\lambda_1 x} = \frac{\lambda_1}{k_0 + k_1 \lambda_1} G(p) \lambda_1 e^{\lambda_1 x}. \tag{14}$$

We obtain again that

$$\left\| \frac{\partial U}{\partial x} \right\|^2_{L^H_{22}} = \int_{\mathbf{R}^+} \left\| \frac{\partial U}{\partial x}(x,p) \right\|^2_{H^2} dx \leq N C_2 \int_{\mathbf{R}^+} \left\| e^{\lambda_1(p)x} G(p) \right\|^2_{H^2} dx$$
$$\leq N C_2 \int_{\mathbf{R}^+} e^{-bx/2} \|G(p)\|^2_{H^2} dx \leq C_3 \|G\|_{H^2}. \tag{15}$$

By (7), $\partial^2 U/\partial x^2$ can be expressed as a linear combination of U, pU, and $\partial U/\partial x$. By (13)-(15),

$$\int_{\mathbf{R}^+} \left\| \frac{\partial^2 U}{\partial x^2}(x,p) \right\|_{H^2}^2 dx \le C_4 \left(\int_{\mathbf{R}^+} \left\| \frac{\partial U}{\partial x}(x,p) \right\|_{H^2}^2 dx + \sum_{m=0,1} \int_{\mathbf{R}^+} \|p^m U(x,p)\|_{H^2}^2 \, dx \right).$$

It follows that

$$\int_{\mathbf{R}^+} \left\| \frac{\partial^2 U}{\partial x^2}(x,p) \right\|_{H^2}^2 dx \le C_5 (\|G\|_{H^2}^2 + \|pG(p)\|_{H^2}^2). \tag{16}$$

Here C_k are constants that depend on a, b, c, k_0, k_1. By (13)-(16), estimate (8) holds.

Let $u(x,\cdot) \triangleq \mathcal{F}^{-1} U(x, i\omega)|_{\omega \in \mathbf{R}}$. By (13), it follows that the corresponding inverse Fourier transforms $u(x,\cdot) = \mathcal{F}^{-1} U(x, i\omega)|_{\omega\in\mathbf{R}}$, $\frac{\partial u}{\partial t}(x,\cdot) = \mathcal{F}^{-1}(pU(x,i\omega)|_{\omega\in\mathbf{R}})$ are well defined and are vanishing for $t < 0$. In addition, we have that $\overline{U(x,i\omega)} = U(x,-i\omega)$ (for instance, $\overline{G_k(i\omega)} = G_k(-i\omega)$, $\overline{e^{x\lambda_k(i\omega)}} = e^{x\lambda_k(-i\omega)}$, etc). It follows that the inverse of Fourier transform $u(x,\cdot) = \mathcal{F}^{-1}U(x,\cdot)$ is real. By (8), estimate (4) holds. Therefore, u is the solution of (1) in $\mathcal{W}$. The uniqueness is ensured by the linearity of the problem, by estimate (4), and by the fact that $\mathcal{L}u(x,\cdot)$, $\mathcal{L}(\partial^k u(x,\cdot)/\partial x^k)$, and $\mathcal{L}(\partial u(x,\cdot/\partial t)$ are well defined on $\mathbf{C}^+$ for any function u from $\mathcal{W}$, i.e, (7) must be satisfied together with (8). This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Rewrite the parabolic equation as the one with c replaced by $c + M$ and $g(t)$ replaced by $g(t)e^{-Mt}$. By Theorem 1, solution $u_M \in \mathcal{M}$ of the new equation exists. Clearly, $u(x,t) = e^{Mt}u_M(x,t)$ is the solution of of the original problem. $\square$

3 Some application

Total absorbing on the boundary

Let $T > 0$ be given. Let us consider the following well-posed boundary value problem on semi-plane:

$$\begin{aligned} &\frac{\partial v}{\partial t}(x,t) + a\frac{\partial^2 v}{\partial x^2}(x,t) + b\frac{\partial v}{\partial x}(x,t) + cv(x,t) = 0, \qquad x > 0,\ t \in [0,T], \\ &v(x,T) \equiv v_*(x), \\ &k_0 v(0,t) + k_1 \frac{\partial v}{\partial x}(0,t) \equiv g(t). \end{aligned} \tag{17}$$

Here $a, b, c, k_0, k_1 \in \mathbf{R}$ are constants such that (2) holds.

Theorem 3 *For any $g \in \Gamma$, there exists $v_* \in L_2(\mathbf{R}^+)$ such that $v(x,0) \equiv 0$, where $v \in \mathcal{W}$ is the solution of well-posed problem (17).*

Proof. It suffices to take the solution $u \in \mathcal{W}$ of problem (1) and take $v_*(x) \triangleq u(x,T)$, $v = u$. $\square$

Note that one can rewrite problem (17) as a well posed problem for forward parabolic equation with initial time at time $t = 0$ via time change $t \to T - t$. In that case, the phenomena described in Theorem 3 looks more impressive.

Restoring past distributions of diffusion processes

Consider the following stochastic process

$$y^x(t) = x + bt + \sigma w(t). \tag{18}$$

Here $x \geq 0$, $w(t)$ is a scalar Wiener process, $b > 0$ and $\sigma > 0$ are constants.

Let a be a random number such that $a \geq 0$ and it has the probability density function $\rho \in L_2(\mathbf{R}^+)$ which is supposed to be unknown. We assume also that a is independent from $w(t) - w(t_1)$ for all $t > t_1 \geq 0$.

Let $y(t) = y^a(t)$ be the solution of Ito equation (18) with the initial condition $y(0) = a$. Set $\tau^a \triangleq \min\{t > 0 : y^a(t) = 0\}$.

Let $\mathbb{I}$ denotes the indicator function of an event.

Let $p(x,t)$ be the probability density function of the process $y^a(t)$ if this process is killed at 0 and inside $(0, +\infty)$ with the rate of killing c (case of $c > 0$ is not excluded for the sake of generality). More precisely, p is such that

$$\int_B p(x,t)dx = \mathbf{E} e^{ct} \mathbb{I}_{(y^a(t) \in B)} \mathbb{I}_{\{\tau^a \geq t\}}$$

for any domain $B \subset \mathbf{R}^+$. It is known that evolution of p is described by the parabolic equation being ajoint to (1) with the boundary value conditions $p(x,0) \equiv \rho(x)$, $p(0,t) \equiv 0$.

For $g \in \Gamma$, let $u_g = u_g(x,t)$ be the solution of the problem (1), where $k_0 = 1$, $k_1 = 0$.

Let $T > 0$ be given, and let $\Psi_g(x) \triangleq u_g(x,T)$.

Theorem 4 *For all functions $g \in \Gamma$ and all $c \in \mathbf{R}$,*

$$\mathbf{E}e^{c\tau^a}g(\tau^a)\mathbb{I}_{\{\tau^a<T\}} = -\mathbf{E}e^{cT}\Psi_g(y(T))\mathbb{I}_{\{\tau^a\geq T\}} = -\int_{\mathbf{R}^+} p(x,T)\Psi_g(x)dx. \qquad (19)$$

Theorem 4 allows to solve effectively the following inverse problem: find the distribution of $\tau^a\mathbb{I}_{\{\tau^a<T\}}$ for unknown distribution of a using the "future" values of $p(x,T)$ only. More precisely, one can find the values of the expectations at the left hand side of (19) for all $g \in \Gamma$ using only $p(x,T)$ via the following algorithm:

(a) Find u_g as the solution of (1);

(b) Find $\Psi_g = u_g(\cdot, T)$;

(c) Using known $p(x,T)$, calculate the integral at the right hand side of (19).

This approach does not require regularization of a ill-posed problem such as in Beck (1985) or Tikhonov and Arsenin (1977).

Proof of Theorem 4. Clearly,

$$\mathbf{E}e^{c(\tau^a\wedge T)}u_g(y(\tau^a\wedge T),\tau^a\wedge T) = \mathbf{E}e^{c\tau^a}g(\tau^a)\mathbb{I}_{\{\tau^a<T\}} + \mathbf{E}e^{cT}\Psi_g(y(T))\mathbb{I}_{\{\tau^a\geq T\}}.$$

By Ito formula,

$$\mathbf{E}e^{c(\tau^a\wedge T)}u_g(y(\tau^a\wedge T),\tau^a\wedge T) = \mathbf{E}u_g(a,0) = 0.$$

In addition,

$$\mathbf{E}e^{cT}\Psi_g(y(T))\mathbb{I}_{\{\tau^a\geq T\}} = \int_{\mathbf{R}^+} p(x,T)\Psi_g(x)dx.$$

Then the result follows. □

References

Beck, J.V. *Inverse Heat Conduction.* 1985. John Wiley and Sons, Inc..

Duren, P. *Theory of H^p-Spaces.* 1970. Academic Press, New York.

Tikhonov, A. N. and Arsenin, V. Y. *Solutions of Ill-posed Problems.* 1977. W. H. Winston, Washington, D. C.